Lab 28

Purpose

To find the roots of functions using Newton's method and the secant method.

28.1 Newton's Method

In previous labs you used both the bisection and decsection methods for locating the roots of functions. Another method for locating the roots of a function is *Newton's Method,* or the *Newton-Raphson* method. This method is commonly used, due to its rapid convergence as compared to other methods.

Newton's method can be derived as follows: For x close to p, we can approximate the derivative of $f(x)$ as

$$f'(x) \approx \frac{f(x) - f(p)}{x - p} \tag{28.1}$$

Using the fact that p is a zero of the function, $f(p) = 0$, and solving for p, we find

$$p \approx x - \frac{f(x)}{f'(x)}. \tag{28.2}$$

We can obtain a sequence of approximations to p by starting with a guess, $x = p_0$ on the right side of equation (28.2) to get a new approximation

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}. \tag{28.3}$$

Now, use this approximation to obtain a better approximation by inserting it on the right side of (28.2):

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}. \tag{28.4}$$

Continuing in this manner, one obtains a sequence of approximations

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}. \tag{28.5}$$

If the sequence $\{p_n\}$ converges, then it converges to p, the root of the given function.

28.2 The Secant Method

Under reasonable assumptions about the function $f(x)$, Newton's method converges. However, it has a major difficulty: In order to apply the method, one needs to be able to compute the derivative of the given function. Sometimes, this may involve more arithmetic operations; for example, if $f(x) = x^2 3^x \cos 2x$, then $f'(x) = 2x3^x \cos 2x + 3x^3 \cos 2x - 2x^2 3^x \sin 2x$. To get around this problem,

one can use the following variation: Since

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}, \qquad (28.6)$$

then we could let $x = p_{n-2}$ to obtain

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}. \qquad (28.7)$$

Substituting this expression for the derivative in Newton's method (6), we obtain

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}. \qquad (28.8)$$

The technique resulting from the use of this iteration is called the *Secant method.*

Instructions

- Now, make up your own routine to carry out Newton's method for the function

  $$f(x) = x^3 + 4x^2 - 10, \qquad x \in [1, 2]. \qquad (28.9)$$

  Obtain the root to five place accuracy. How many steps did it take, as compared to the bisection method? [Check your notes on the Bisection lab.]
- Carry out the process a little further to get 8 place accuracy. Save your worksheet.
- Use the secant method to determine the roots of (28.9) to eight place accuracy. Save your worksheet.

Exercise

▷ **Exercise 28.1** A can in the shape of a right cylinder is to be constructed to contain 1000 cm^3.[1] The circular top and bottom of the can must have a radius 0.25 cm more than the radius of the can so that the excess can be used to form a seal with the side. The sheet of material being formed into the side of the can must also be 0.25 cm longer than the circumference of the can so that the seal can be formed. Find, to within 10^{-4}, the minimal amount of material needed to construct the can.

[1] Adapted from *Numerical Analysis,* R.L. Burden and J.D. Faires, PWS-Kent, p 82 (1989)