Lab 8

Straightening Quadratic Functions

Purpose

To fit data satisfying a quadratic relationship.

8.1 Quadratic Functions

Most scientific laws can be expressed as relations between two, or more, physical quantities. When two quantities are related such that one is uniquely given in terms of the other, we say that a *functional relationship* exists between them. Once a functional relation is established between physical quantities, one can formulate an empirical law. These laws can then be verified by further experiments, or used to predict the results of experiments which can be carried out for other ranges of the physical quantities.

One can discover an empirical law by seeking a curve that *best fits* the observed data; i.e., one finds a curve that deviates the least from the data points[1]. The simplest laws to obtain are linear relations. However, the more interesting laws are not linear in nature. We often find that data can be described by quadratic, power, or exponential laws.

In this lab we will see how to use the linear fit to study the familiar quadratic function. The general form of this type of function is given by

$$f(x) = ax^2 + bx + c. \tag{8.1}$$

The graph of the function $y = f(x)$ is known as a *parabola.* When $a > 0$ the function has a minimum value, while for $a < 0$ the function attains a maximum value. The point at which these extrema occur is called the *vertex,* or *turning point,* of the function. Its coordinates are given in terms of the coefficients in (8.1) as

$$(-\frac{b}{2a}, f(-\frac{b}{2a})). \tag{8.2}$$

Figure 8.1: Quadratic

The roots of (8.1) are the values of x that make y vanish. These can easily be determined from the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{8.3}$$

An example of a physical law, which is quadratic in nature, is that of the distance as a function of time for free fall. From Newton's laws of motion, we have that

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[1] The method of nonlinear regression, which is a generalization of the linear regression, we will not covered in this course.

$$h = -\frac{1}{2}gt^2 + v_0 t + h_0, \tag{8.4}$$

where v_0 is the initial vertical velocity, g is the acceleration due to gravity (approximately 9.81 m/s^2), and h is the distance above the ground with h_0 the position at time $t = 0$.

The roots of this equation are easily determined from (8.3) and the maximum height will be reached at time $t_m = v_0/g$.

We now perform a free fall experiment, assuming that the relation in (8.4) holds.

We gather pairs of data (t, h) and assume that we know the value of h_0. Plotting h vs. t will lead to a parabolic set of points. However, we can rearrange equation (8.4) to form a linear relationship. Namely, by subtracting h_0 from both sides of (8.4) and dividing by t, we obtain

$$\frac{h - h_0}{t} = -\frac{g}{2}t + v_0. \tag{8.5}$$

Figure 8.2: Best Fit Line

A plot of $(h - h_0)/t$ versus t will yield a straight line with an intercept of v_0 and a slope of $-g/2$. [**Note:** You cannot compute the left side of this equation for $t = 0$. Why?]

8.2 Interpolation

Figure 8.3: Interpolation

Having determined these values, we could see how well equation (8.4) fits the original data. Using this approximate curve, we could *interpolate,* or *extrapolate* the data. By interpolation, we mean that we could determine the height at a given time during the experiment. We could also extrapolate the data to obtain heights for values outside of the range of times considered. Linear interpolation proceeds as follows: Assume that we have two values of the independent variable, x_1 and x_2, and that we have the corresponding values of the dependent variable, y_1 and y_2. We would like to find the value y, which corresponds to an intermediate value, x, which lies between x_1 and x_2. From the figure to the right, we note that the ratio of $x - x_1$ to the full length of the interval $x_2 - x_1$ is the same as the ratio of $y - y_1$ to $y_2 - y_1$. Therefore,

$$y - y_1 = \frac{x - x_1}{x_2 - x_1}(y_2 - y_1) \tag{8.6}$$

Solving for y, we obtain

$$y = y_1 + \frac{x - x_1}{x_2 - x_1}(y_2 - y_1) \tag{8.7}$$

Therefore, equation (8.7) can be used to get values of the dependent variable for intermediate values of the independent variable. In fact, this formula is also useful for doing linear extrapolation, in which case we are interested in values outside of the given range of independent variables.

Instructions

In the following problem [2] one can employ the above ideas to study the given data. Create a MathCAD worksheet for this problem. Provide data tables and graphs in the worksheet and print out the final results for your notebook.

In order to see how quickly the heart rate returns to normal after running, four reasonably fit students exercised on a tread mill for about 12 minutes until their heart rates reached 190 beats per minute. They then stopped exercising and their heart rate was monitored at regular intervals as it returned to normal. The average R of the four heart rates as a function of time after exercise was as follows:

t min	1	3	5	7	9	11
R beats/min	170	136	111	95	86	80

- Enter the data into appropriate arrays, say **R** and **t** and plot this data.
- Assume that this data over the time interval of the experiment can be described by a model of the form
$$R = 190 + at + bt^2, \tag{8.8}$$
where a and b are constants. Determine the type of plot that is needed to straighten this function.
- Define new variables in terms of the data based on your work in the last step. Plot this data. Do you have a linear relationship?
- Find the best fit line through these points and plot it with the data.

Exercises

▷ **Exercise 8.1** From your graph obtain values for a and b.

▷ **Exercise 8.2** Use these values to fit the original data. What is the largest deviation of your best fit quadratic function from the data?

▷ **Exercise 8.3** Determine the time indicated by the model for the heart rate R to become a minimum. What is this minimum predicted to be?

▷ **Exercise 8.4** Use linear interpolation to find the heart beat rate after 10 minutes. Compare this result with the value obtained from the empirical model given by (8.8).

▷ **Exercise 8.5** Extrapolate your data to determine the rate after 15 minutes. Compare this result with your empirical law (8.8).

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[2] Adapted from a problem in *Introductory Mathematics Through Science Applications*, J. Berry, A. Norcliffe, and S. Humble, Cambridge Univ. Press, 1989, p. 32