Lab 29

Purpose

To become familiar with the process of iteration and its convergence properties.

29.1 Fixed Point Iterations

Newton's method and the secant method are examples of *fixed-point iterations.* In these methods one seeks the solution of the equation

$$g(x) = x. \tag{29.1}$$

The solution to such an equation is called a *fixed point.* In fact, if we wanted to find the root of $f(x) = 0$, we could search for the fixed points of $g(x) = x$, where $g(x) \equiv x - f(x)$. You should convince yourself of this!

There are many ways to change a given equation, $f(x) = 0$, into a fixed point equation of the form (29.1). For example, if one is interested in obtaining the roots of the equation

$$x^3 + 4x^2 - 10 = 0, \tag{29.2}$$

we could change the equation as follows:

$$\begin{aligned} 4x^2 &= 10 - x^3 \\ x^2 &= \frac{1}{4}(10 - x^3) \\ x &= \pm\frac{1}{2}(10 - x^3)^{1/2}. \qquad (29.3) \end{aligned}$$

To obtain a positive solution, one chooses the positive sign.

Several ways to manipulate equation (29.2) into the form of a fixed point equation are given below:

$$\begin{aligned} x &= x - x^3 - 4x^2 + 10 \\ x &= \frac{1}{2}(10 - x^3)^{1/2} \\ x &= \left(\frac{10}{4 + x}\right)^{1/2} \end{aligned}$$

$$x = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}. \tag{29.4}$$

You should verify that the fixed point of each of these equations is a solution of the original equation.

The procedure for determining the fixed point is given by the following theorem:
Fixed Point Theorem: *Let $g(x)$ be a continuous function on $[a, b]$ such that $g(x) \in [a, b]$ for all x in $[a, b]$. Further, suppose that $g'(x)$ exists on (a, b) with*

$$|g'(x)| \leq k < 1 \qquad \text{for all } x \in (a, b). \tag{29.5}$$

If p_0 is any number in $[a, b]$, then the sequence defined by

$$p_n = g(p_{n-1}), \qquad n \geq 1, \tag{29.6}$$

converges to the unique fixed point p in $[a, b]$.

Therefore, equation (29.6) defines the sequence of approximations to the fixed point of the equation $g(x) = x$, which can be used to obtain the roots of our given function. The above figure shows an example of how this process takes place. Superimposed on the graph $y = g(x)$ is a graph of $y = x$ These intersect at the point (p, p). Now, starting with a guess of x_0, as a crude approximation to p, one obtains from the iterative scheme (29.6) that $x_1 = g(x_0)$. This is obtained graphically by moving along the line $y = x_1$ until it crosses $y = x$. The process is now repeated, using the new guess, x_1. **Note:** We can only get convergence if condition (29.5) is satisfied. Furthermore, the smaller k is, the faster the convergence.

Instructions

- Create a generic worksheet for carrying out the fixed point iteration (29.6).
- Now, apply this to each equation in (29.4) to determine the roots of (29.2).
- Consider the iterative scheme

  $$x_{n+1} = \frac{\lambda x_n + 1 - \sin x_n}{1 + \lambda} \tag{29.7}$$

  for finding the root of $f(x) = 1 - x - \sin x$, which is near $x = 0.5$. Using MathCAD, search for the constant λ, which gives a rapidly convergent scheme.

Exercises

▷ **Exercise 29.1** Show that the fixed points of the equations in (29.4) are solutions of equation (29.2). Determine the conditions for convergence for the equations in (29.4), using the condition in (29.5).

▷ **Exercise 29.2** Which of the methods converged? To what value does each converge?

▷ **Exercise 29.3** Which method did not converge and why?

▹ **Exercise 29.4** Which converges the fastest? Why do they converge at different rates? What governs this rate of convergence?

▹ **Exercise 29.5** Using the criterion of the fixed point theorem (29.5), analytically determine the value of λ, which gives the best convergence in (29.7).