

## L a b 2 4

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### Purpose

To use probability densities to compute the average molecular speed in an ideal gas.

### 24.1 Average Molecular Speeds

Improper integrals of the form

$$\int_0^{\infty} f(x) dx \quad (24.1)$$

occur frequently in applications. A typical example comes from the kinetic theory of gases and involves concepts from continuous probability. A result from this theory is the law of distribution of molecular velocities, describing how many molecules out of each mole of gas have molecular velocities in any selected range of values. In particular, the law tells us what the probability is for the velocity of a single molecule to lie in a given range. This is referred to as the *Maxwellian distribution*, named after James Clerk Maxwell, who was one of the first to derive this law<sup>1</sup>.

The probability  $P(v)$  that a molecule has a speed between  $v$  and  $v + dv$  is given as

$$P(v) = \frac{4}{\alpha^3 \pi^{1/2}} v^2 e^{-v^2/\alpha^2} dv. \quad (24.2)$$

It is common to define the *probability density*,  $f(v)$ , through the expression  $P(v) = f(v) dv$ . From equation (24.2),

$$f(v) = \frac{4}{\alpha^3 \pi^{1/2}} v^2 e^{-v^2/\alpha^2}. \quad (24.3)$$

For an ideal gas of molecules with molecular weight  $M$  at temperature  $T$ ,

$$\alpha^2 = \frac{2RT}{M}, \quad (24.4)$$

where  $R = 8.314 \text{ J/mol}\cdot\text{K}$  is the gas constant.

For a continuous probability distribution, the chance that  $X$  falls in between  $a$  and  $b$  can be computed by integrating the density:

$$P(a \leq X \leq b) = \int_a^b f(x) dx, \quad (24.5)$$

where  $f(x) \geq 0$ . If one includes all possible outcomes, the probability is necessarily one (i.e., the event is 100% likely):

$$P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f(x) dx = 1. \quad (24.6)$$

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<sup>1</sup>For more in depth coverage, see *Thermal Physics*, C. Kittel, Wiley & Sons, Ch. 13, (1969)

The probability distribution can also be used to determine the average, or mean, value of  $x$ . This expected value of  $x$  is found as

$$\bar{x} = \int_{-\infty}^{\infty} x f(x) dx. \quad (24.7)$$

Using the Maxwellian distribution, one can find the average speed of the molecules in an ideal gas, which is at some given temperature  $T$ , by evaluating the integral

$$\bar{v} = \int_0^{\infty} v f(v) dv = \int_0^{\infty} \frac{4}{\alpha^3 \pi^{1/2}} v^3 e^{-v^2/\alpha^2} dv. \quad (24.8)$$

This integral can be computed exactly to give

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}. \quad (24.9)$$

It is common to instead compute the root-mean-square velocity,  $v_{\text{rms}}$ ,

$$v_{\text{rms}}^2 = \int_0^{\infty} v^2 f(v) dv = \int_0^{\infty} \frac{4}{\alpha^3 \pi^{1/2}} v^4 e^{-v^2/\alpha^2} dv. \quad (24.10)$$

The last integral can be evaluated, yielding

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}. \quad (24.11)$$

In equations (24.8) and (24.10) the resulting integrals are both improper integrals, since each has one infinite limit. In this lab you will compute these integrals and compare your results to the exact values found in (24.9) and (24.11).

## Instructions

- Consider an ideal gas of oxygen molecules,  $M = 32 \times 10^{-3}$  kg/mol, which is at 300°K. Using equation (24.8), determine the average speed of an oxygen molecule.
- For the same oxygen gas, compute the root-mean-square velocity, which is given by equation (24.10).

## Exercises

- ▷ **Exercise 24.1** Compare your results to the speeds given in equations (24.9) and (24.11). What is the percent error in your computations?
- ▷ **Exercise 24.2** Compute the ratio of the average speed to the root-mean-square speed. How does this compare with the theoretical ratio?
- ▷ **Exercise 24.3** How many feet does this molecule travel in one second? What is its speed in miles per hour?
- ▷ **Exercise 24.4** The complicated integrals in equations (24.8) and (24.10) can be evaluated using integration by parts, the method of substitution and  $\int_0^i nfty e^{-\beta v^2} dv = \frac{1}{2} \sqrt{\frac{\pi}{\beta}}$ . Show that  $\int_0^i nfty v^3 e^{-\beta v^2} dv = \frac{1}{2\beta^2}$  and use this result to verify equation (24.8).