Lab 27

Purpose

Purpose: To study lengths and areas of polar curves.

27.1 Introduction

In the last lab you experimented with a variety of polar curves. Now you will look at some of these curves and determine their lengths and the areas of particular regions bounded by these curves.
Consider the curve $r = f(\theta)$ for $a \le \theta \le b$. The enclosed area can be computed by partitioning the $\theta-$range into subangles such as the one depicted in the figure to the right. In this figure the area corresponding to $d\theta$ can be approximated by the area subtended by a central angle of $d\theta$ and a radius r. This area is found to be

$$dA = \frac{1}{2} r^2 d\theta. \tag{27.1}$$

Adding up all such areas over the partition, yields the total area from $\theta = a$ to $\theta = b$

$$A[a,b] = \int_a^b \frac{1}{2} r^2 d\theta. \tag{27.2}$$

The length of a curve can be obtained in terms of the differential $ds = \sqrt{dx^2 + dy^2}$. The total length of an arc is then given by

$$L = \int_{s(a)}^{s(b)} ds. \tag{27.3}$$

For polar coordinates this differential is easily computed. Recalling the relation between polar and Cartesian coordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{27.4}$$

we can evaluate the differentials

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta \tag{27.5}$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta. \tag{27.6}$$

This gives the differential of arclength as

$$ds = \sqrt{r^2 d\theta^2 + dr^2} = \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta. \tag{27.7}$$

Therefore, the length of a polar curve $r = f(\theta)$ over the interval $[a, b]$ is

$$L[a, b] = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta. \tag{27.8}$$

For the cardioid, $r(\theta) = 1 - \cos(\theta)$ from $\theta = 0$ to $\theta = 2\pi$, the corresponding area and length can be computed as

$$A[0, 2\pi] = \frac{1}{2}\int_0^{2\pi} (1 - \cos(\theta))^2 d\theta = \frac{3\pi}{2}, \tag{27.9}$$

$$L[0, 2\pi] = \int_0^{2\pi} \sqrt{(1 - \cos(\theta))^2 + \sin^2\theta} = 8. \tag{27.10}$$

27.2 Exercises

In each of the exercises below you should follow these steps:

- Enter the function
- Graph the function $r(\theta)$ over an appropriate interval to see the polar curve.
- Determine the appropriate range of θ values for performing the computation of the desired area and/or arclength.
- Compute the area and/or arclength numerically.

▷ **Exercise 27.1** Consider the limacon $r = 1 + 2\cos\theta$. Find the area of the inner loop. What is the area between the two loops? Compute the entire length of the limacon.

▷ **Exercise 27.2** Consider the four-leaf rose $r = \sin(2\theta)$. Find the area of the leaf in the first quadrant. What is its length?

▷ **Exercise 27.3** Find the total area of the 6 leaf rose $r^2 = 2\sin 3\theta$.

▷ **Exercise 27.4** Find the length of the spiral given by $r = \theta^2$ with $0 \leq \theta \leq \sqrt{5}$.

▷ **Exercise 27.5** Determine the area, which is shared by the two cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$ [Hint: You should use the symmetry in the region of interest. Does your result make sense?]

▷ **Exercise 27.6** Verify the results in equations (27.9)-(27.10) using the methods of integration which you have learned.